Lecture 4
Binary search (cont.), insertion/selection sort, analysis of quick sort

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Binary Search: Searching in a sorted array

- Input is a sorted array A and an item x.
- Problem is to locate x in the array.

- We will show that binary search is an optimal algorithm for solving this problem.

Binary Search: Searching in a sorted array

Input:	A:	Sorted array with n entries $[0..n-1]$
	x:	Item we are seeking

Binary Search: Searching in a sorted array

Input:	A:	Sorted array with n entries $[0..n-1]$
	x:	Item we are seeking

Output: Location of x, if x found
-1, if x not found

Binary Search: Searching in a sorted array

Input: A: Sorted array with n entries $[0..n-1]$
x: Item we are seeking

Output: Location of x, if x found
-1, if x not found

```
def binarySearch(A,x,first,last)
if first > last:
  return (-1)
else:
  mid = ⌊(first+last)/2⌋
  if x == A[mid]:
    return mid
  else if x < A[mid]:
    return binarySearch(A,x,first,mid-1)
  else:
    return binarySearch(A,x,mid+1,last)

binarySearch(A,x,0,n-1)
```

Binary Search: Analysis of Running Time (continued)

- Binary search in an array of size 1: 1 decision
- Binary search in an array of size $n > 1$: after 1 decision, either we are done, or the problem is reduced to binary search in a subarray with a worst-case size of $\lfloor n/2 \rfloor$
- So the worst-case time to do binary search on an array of size n is $T(n)$, where $T(n)$ satisfies the equation

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & \text{otherwise} \end{cases}$$

- The solution to this equation is:

$$T(n) = \lfloor \lg n \rfloor + 1$$

  This can be proved by induction.
- So binary search does $\lfloor \lg n \rfloor + 1$ 3-way comparisons on an array of size n, in the worst case.

Optimality of binary search

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.
- Our lower bound is established using a Decision Tree model.

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.
- Our lower bound is established using a Decision Tree model.
- Note that the bound is exact (not just asymptotic)

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.
- Our lower bound is established using a Decision Tree model.
- Note that the bound is exact (not just asymptotic)
- Our lower bound is on the worst case

Optimality of binary search

- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.
- Our lower bound is established using a Decision Tree model.
- Note that the bound is exact (not just asymptotic)
- Our lower bound is on the worst case
  - It says: for every algorithm for finding an item in an array of size n, there is some input that forces it to perform $\lfloor \lg n \rfloor + 1$ comparisons.

Optimality of binary search

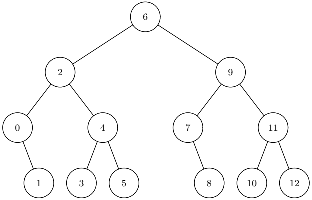
- We will establish a lower bound on the worst-case number of decisions required to find an item in an array, using only 3-way comparisons of the item against array entries.
- The lower bound we will establish is $\lfloor \lg n \rfloor + 1$ 3-way comparisons.
- Since Binary Search performs within this bound, it is optimal.
- Our lower bound is established using a Decision Tree model.
- Note that the bound is exact (not just asymptotic)
- Our lower bound is on the worst case
  - It says: for every algorithm for finding an item in an array of size n, there is some input that forces it to perform $\lfloor \lg n \rfloor + 1$ comparisons.
  - It does not say: for every algorithm for finding an item in an array of size n, every input forces it to perform $\lfloor \lg n \rfloor + 1$ comparisons.

The decision tree model for searching in an array

The decision tree model for searching in an array

Consider any algorithm that searches for an item x in an array A of size n by comparing entries in A against x. Any such algorithm can be modeled as a decision tree:

Example: Decision tree for binary search with $n = 13$:

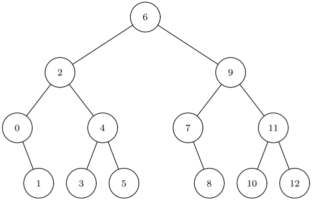


The decision tree model for searching in an array

Consider any algorithm that searches for an item x in an array A of size n by comparing entries in A against x. Any such algorithm can be modeled as a decision tree:

- Each node is labeled with an integer $\in \{0 \ldots n-1\}$.

Example: Decision tree for binary search with $n = 13$:

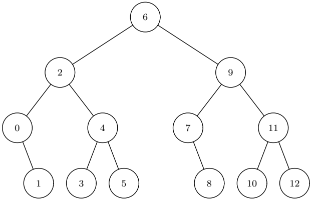


The decision tree model for searching in an array

Consider any algorithm that searches for an item x in an array A of size n by comparing entries in A against x. Any such algorithm can be modeled as a decision tree:

- Each node is labeled with an integer $\in \{0 \ldots n-1\}$.
- A node labeled i represents a 3-way comparison between x and $A[i]$.

Example: Decision tree for binary search with $n = 13$:

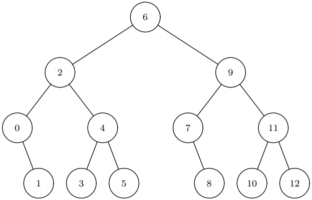


The decision tree model for searching in an array

Consider any algorithm that searches for an item x in an array A of size n by comparing entries in A against x. Any such algorithm can be modeled as a decision tree:

- Each node is labeled with an integer $\in \{0 \ldots n-1\}$.
- A node labeled i represents a 3-way comparison between x and $A[i]$.
- The left subtree of a node labeled i describes the decision tree for what happens if $x < A[i]$.

Example: Decision tree for binary search with $n = 13$:

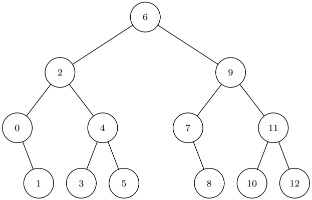


The decision tree model for searching in an array

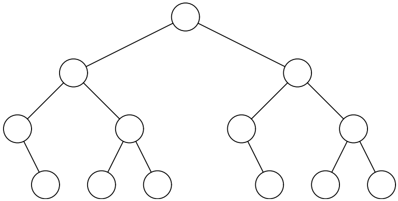
Consider any algorithm that searches for an item x in an array A of size n by comparing entries in A against x. Any such algorithm can be modeled as a decision tree:

- Each node is labeled with an integer $\in \{0 \ldots n-1\}$.
- A node labeled i represents a 3-way comparison between x and $A[i]$.
- The left subtree of a node labeled i describes the decision tree for what happens if $x < A[i]$.
- The right subtree of a node labeled i describes the decision tree for what happens if $x > A[i]$.

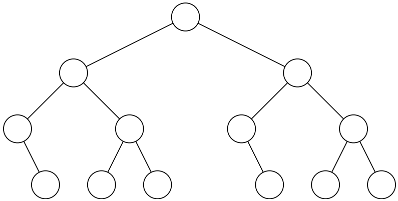
Example: Decision tree for binary search with $n = 13$:



Lower bound on locating an item in an array of size n

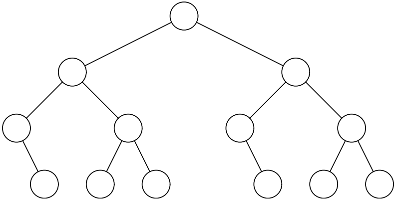


Lower bound on locating an item in an array of size n



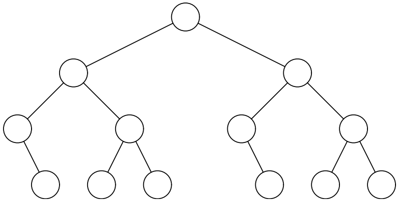
1. Any algorithm for searching an array of size n can be modeled by a decision tree with at least n nodes.

Lower bound on locating an item in an array of size n



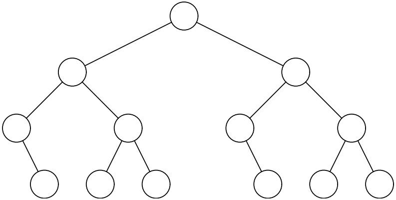
1. Any algorithm for searching an array of size n can be modeled by a decision tree with at least n nodes.
2. Since the decision tree is a binary tree with n nodes, the depth is at least $\lfloor \lg n \rfloor$.

Lower bound on locating an item in an array of size n



1. Any algorithm for searching an array of size n can be modeled by a decision tree with at least n nodes.
2. Since the decision tree is a binary tree with n nodes, the depth is at least $\lfloor \lg n \rfloor$.
3. The worst-case number of comparisons for the algorithm is the depth of the decision tree $+1$. (Remember, root has depth 0).

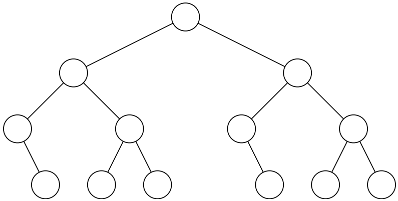
Lower bound on locating an item in an array of size n



1. Any algorithm for searching an array of size n can be modeled by a decision tree with at least n nodes.
2. Since the decision tree is a binary tree with n nodes, the depth is at least $\lfloor \lg n \rfloor$.
3. The worst-case number of comparisons for the algorithm is the depth of the decision tree $+1$. (Remember, root has depth 0).

Hence any algorithm for locating an item in an array of size n using only comparisons must perform at least $\lfloor \lg n \rfloor + 1$ comparisons in the worst case.

Lower bound on locating an item in an array of size n



1. Any algorithm for searching an array of size n can be modeled by a decision tree with at least n nodes.
2. Since the decision tree is a binary tree with n nodes, the depth is at least $\lfloor \lg n \rfloor$.
3. The worst-case number of comparisons for the algorithm is the depth of the decision tree $+1$. (Remember, root has depth 0).

Hence any algorithm for locating an item in an array of size n using only comparisons must perform at least $\lfloor \lg n \rfloor + 1$ comparisons in the worst case.

So binary search is optimal with respect to worst-case performance.

Sorting

Sorting

- Rearranging a list of items in nondescending order.

Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search)

Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search)
- Important step in other algorithms

Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search)
- Important step in other algorithms
- Illustrates more general algorithmic techniques

Sorting

- Rearranging a list of items in nondescending order.
- Useful preprocessing step (e.g., for binary search)
- Important step in other algorithms
- Illustrates more general algorithmic techniques

We will discuss in the class

- Comparison-based sorting algorithms (Insertion sort, Selection Sort, Quicksort, Mergesort, Heapsort)
- Bucket-based sorting methods

Comparison-based sorting

- Basic operation: compare two items.

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings,

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.
- Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.
- Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.
- Measure of time: number of comparisons

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.
- Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.
- Measure of time: number of comparisons
  - Consistent with philosophy of counting basic operations, discussed earlier.

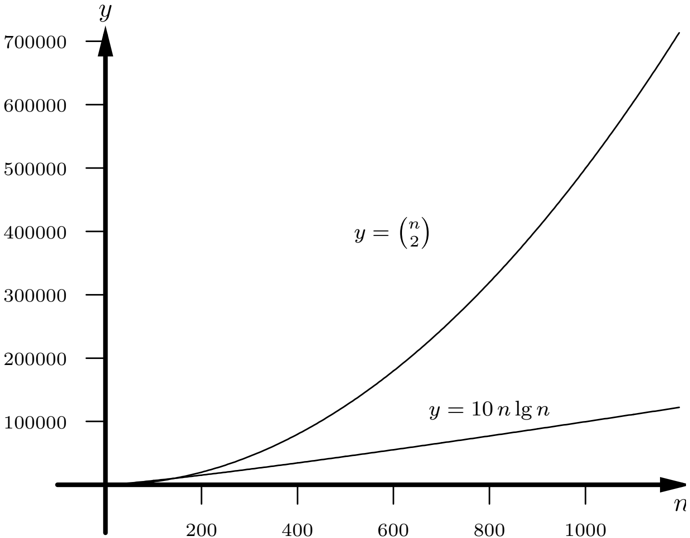
Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.
- Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.
- Measure of time: number of comparisons
  - Consistent with philosophy of counting basic operations, discussed earlier.
  - Misleading if other operations dominate (e.g., if we sort by moving items around without comparing them)

Comparison-based sorting

- Basic operation: compare two items.
- Abstract model.
- Advantage: doesn't use specific properties of the data items. So same algorithm can be used for sorting integers, strings, etc.
- Disadvantage: under certain circumstances, specific properties of the data item can speed up the sorting process.
- Measure of time: number of comparisons
  - Consistent with philosophy of counting basic operations, discussed earlier.
  - Misleading if other operations dominate (e.g., if we sort by moving items around without comparing them)
- Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)

$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work



Some terminology

Some terminology

- A permutation of a sequence of items is a reordering of the sequence. A sequence of n items has $n!$ distinct permutations.

Some terminology

- A permutation of a sequence of items is a reordering of the sequence. A sequence of n items has $n!$ distinct permutations.
- Note: Sorting is the problem of finding a particular distinguished permutation of a list.

Some terminology

- A permutation of a sequence of items is a reordering of the sequence. A sequence of n items has $n!$ distinct permutations.
- Note: Sorting is the problem of finding a particular distinguished permutation of a list.
- An inversion in a sequence or list is a pair of items such that the larger one precedes the smaller one.

Some terminology

- A permutation of a sequence of items is a reordering of the sequence. A sequence of n items has $n!$ distinct permutations.
- Note: Sorting is the problem of finding a particular distinguished permutation of a list.
- An inversion in a sequence or list is a pair of items such that the larger one precedes the smaller one.

Example: The list

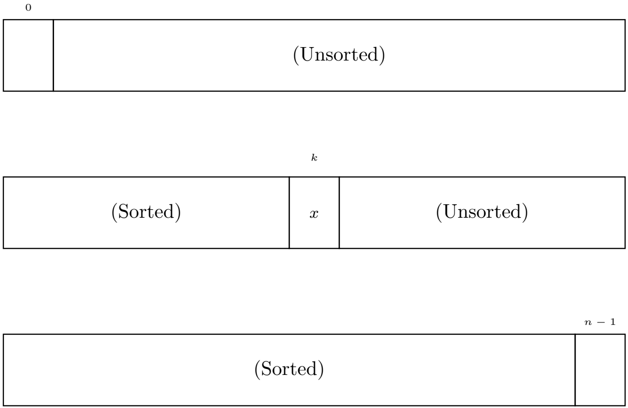
$$18 \quad 29 \quad 12 \quad 15 \quad 32 \quad 10$$

has 9 inversions:

$$\{(18,12), (18,15), (18,10), (29,12), (29,15), (29,10), (12,10), (15,10), (32,10)\}$$

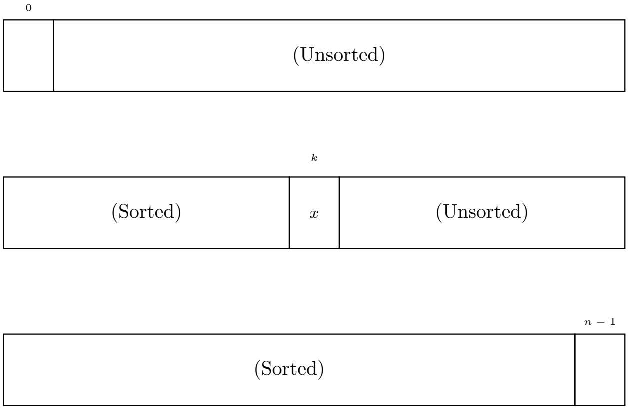
Insertion sort

Insertion sort



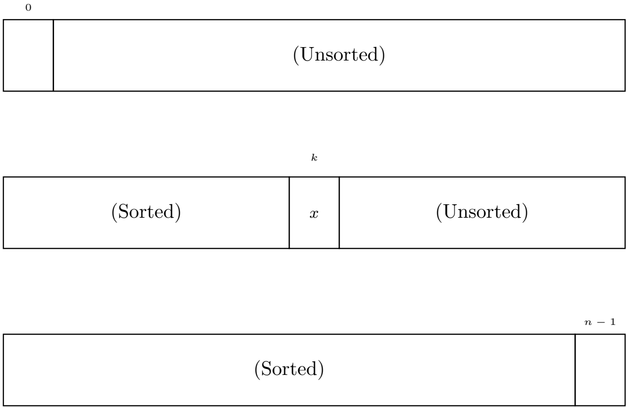
Insertion sort

- Work from left to right across array

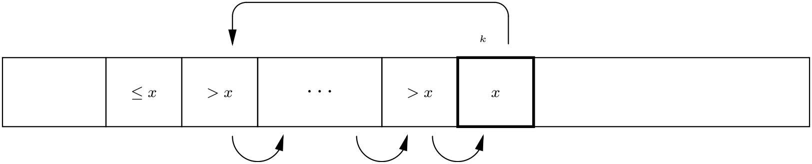


Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left



Insertion sort pseudocode




```
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            A[j+1] = A[j]
            j = j-1
        A[j+1] = x
```

Insertion sort example

23	19	42	17	85	38

23	19	42	17	85	38

19	23	42	17	85	38

19	23	42	17	85	38

17	19	23	42	85	38

17	19	23	42	85	38

17	19	23	38	42	85

Analysis of Insertion Sort

- Worst-case running time:

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1]$, $A[k-2]$, $\ldots$, $A[0]$.

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1], A[k-2], \ldots, A[0]$.
  - Total number comparisons over all iterations is at most:

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = O(n^2).$$

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1]$, $A[k-2]$, ..., $A[0]$.
  - Total number comparisons over all iterations is at most:

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = O(n^2).$$

  - Insertion Sort is a bad choice when n is large. ($O(n^2)$ vs. $O(n \log n)$).

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1]$, $A[k-2]$, ..., $A[0]$.
  - Total number comparisons over all iterations is at most:

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = O(n^2).$$

  - Insertion Sort is a bad choice when n is large. ($O(n^2)$ vs. $O(n \log n)$).
  - Insertion Sort is a good choice when n is small. (Constant hidden in the "big oh" is small).

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1], A[k-2], \ldots, A[0]$.
  - Total number comparisons over all iterations is at most:

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = O(n^2).$$

  - Insertion Sort is a bad choice when n is large. ($O(n^2)$ vs. $O(n \log n)$).
  - Insertion Sort is a good choice when n is small. (Constant hidden in the "big oh" is small).
  - Insertion Sort is efficient if the input is "almost sorted":

$$\text{Time} \leq n - 1 + (\#\ \text{inversions})$$

Analysis of Insertion Sort

- Worst-case running time:
  - On kth iteration of outer loop, element $A[k]$ is compared with at most k elements:
    $A[k-1], A[k-2], \ldots, A[0]$.
  - Total number comparisons over all iterations is at most:

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = O(n^2).$$

  - Insertion Sort is a bad choice when n is large. ($O(n^2)$ vs. $O(n \log n)$).
  - Insertion Sort is a good choice when n is small. (Constant hidden in the "big oh" is small).
  - Insertion Sort is efficient if the input is "almost sorted":

$$\text{Time} \leq n - 1 + (\#\text{ inversions})$$

- Storage: in place: $O(1)$ extra storage

Selection Sort

Selection Sort

- Two variants:

Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order

Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order
  2. Repeatedly (for i from $n-1$ down to 1)

Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order
  2. Repeatedly (for i from $n-1$ down to 1)
     - Find the maximum of $A[0]$,$A[1]$,...,$A[i]$.

Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order
  2. Repeatedly (for i from $n-1$ down to 1)
     - Find the maximum of $A[0]$,$A[1]$,...,$A[i]$.
     - Swap this value with $A[i]$ (no-op if it is already $A[i]$).

Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order
  2. Repeatedly (for i from $n-1$ down to 1)
     - Find the maximum of $A[0]$,$A[1]$,...,$A[i]$.
     - Swap this value with $A[i]$ (no-op if it is already $A[i]$).
- Both variants run in $O(n^2)$ time if we use the straightforward approach to finding the maximum/minimum.

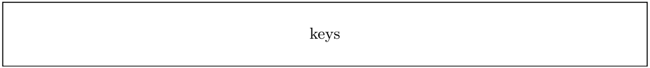
Selection Sort

- Two variants:
  1. Repeatedly (for i from 0 to $n-1$) find the minimum value, output it, delete it.
     - Values are output in sorted order
  2. Repeatedly (for i from $n-1$ down to 1)
     - Find the maximum of $A[0]$,$A[1]$,...,$A[i]$.
     - Swap this value with $A[i]$ (no-op if it is already $A[i]$).
- Both variants run in $O(n^2)$ time if we use the straightforward approach to finding the maximum/minimum.

Quicksort

Quicksort

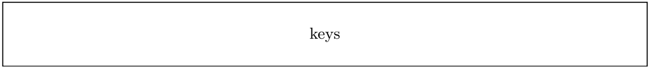
Basic idea



Quicksort

Basic idea

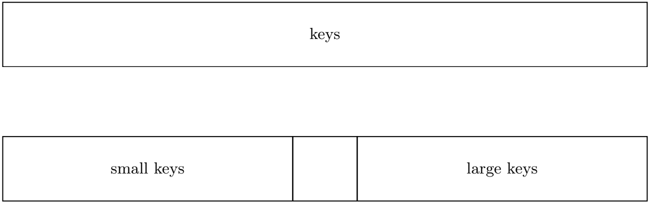
- Classify keys as small keys or large keys. All small keys are less than all large keys



Quicksort

Basic idea

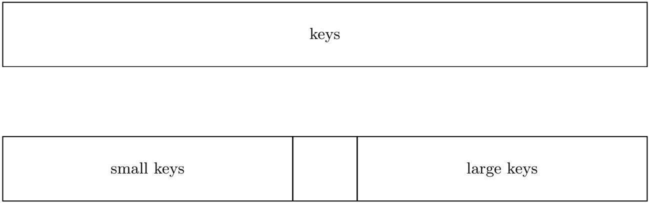
- Classify keys as small keys or large keys. All small keys are less than all large keys
- Rearrange keys so small keys precede all large keys.



Quicksort

Basic idea

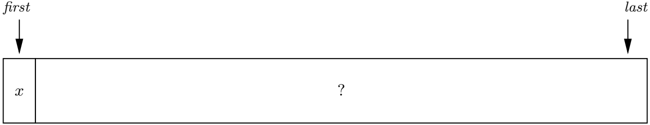
- Classify keys as small keys or large keys. All small keys are less than all large keys
- Rearrange keys so small keys precede all large keys.
- Recursively sort small keys, recursively sort large keys.



Quicksort: One specific implementation

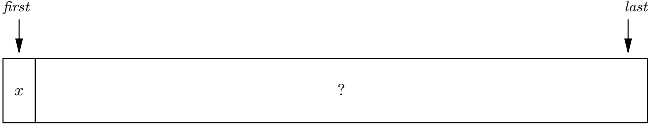
Quicksort: One specific implementation

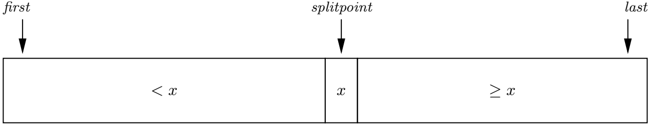
- Let the first item in the array be the pivot value x (also call the split value).



Quicksort: One specific implementation

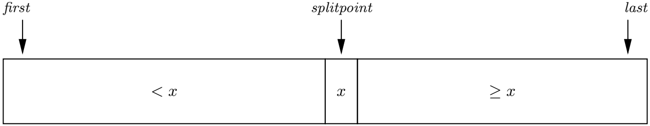
- Let the first item in the array be the pivot value x (also call the split value).
  - Small keys are the keys $< x$.
  - Large keys are the keys $\geq x$.





Pseudocode for Quicksort

```
def quickSort(A,first,last):
    if first < last:
        splitpoint = split(A,first,last)
        quickSort(A,first,splitpoint-1)
        quickSort(A,splitpoint+1,last)
```



The split step

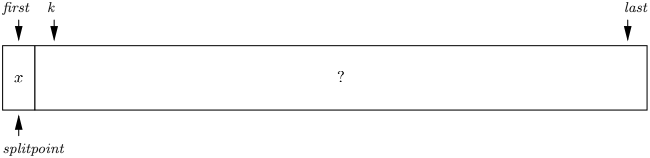
```
def split(A,first,last):
    splitpoint = first
    x = A[first]
    for k = first+1 to last do:
        if A[k] < x:
            A[splitpoint+1] ↔ A[k]
            splitpoint = splitpoint + 1
    A[first] ↔ A[splitpoint]
    return splitpoint
```

Loop invariants:

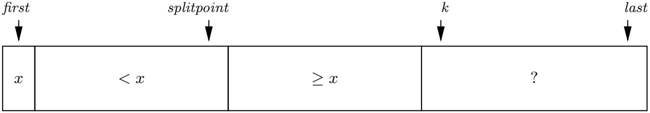
- `A[first+1..splitpoint]` contains keys $< x$.
- `A[splitpoint+1..k-1]` contains keys $\geq x$.
- `A[k..last]` contains unprocessed keys.

The split step

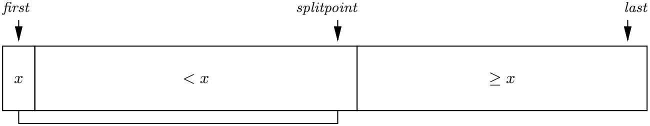
At start:



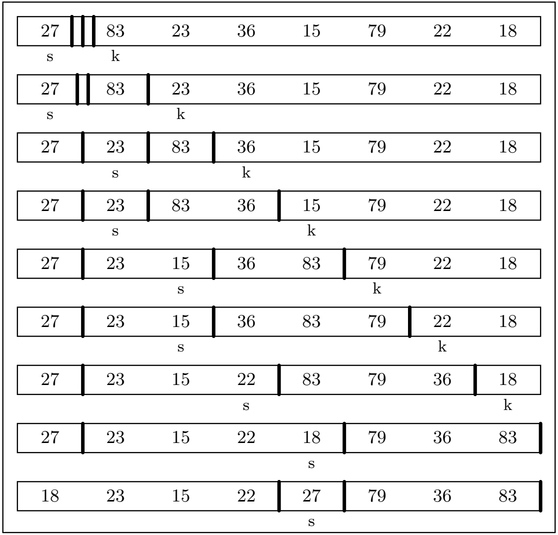
In middle:



At end:



Example of split step



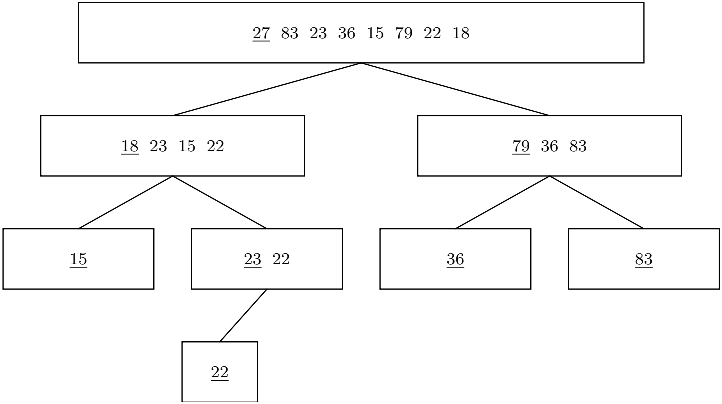
Analysis of Quicksort

Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.

Analysis of Quicksort

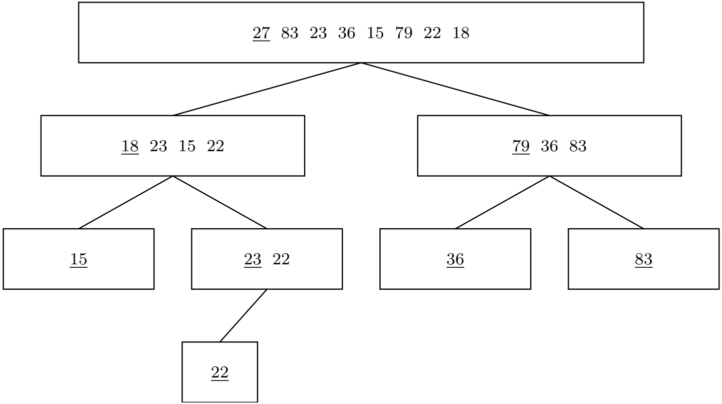
We can visualize the lists sorted by quicksort as a binary tree.



Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.

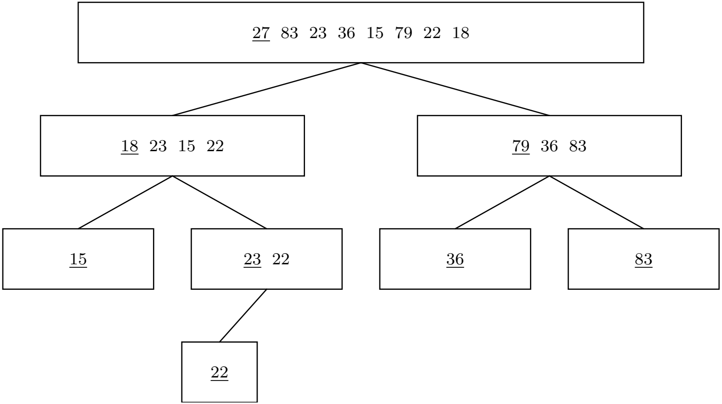
- The root is the top-level list (of all items to be sorted)



Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.

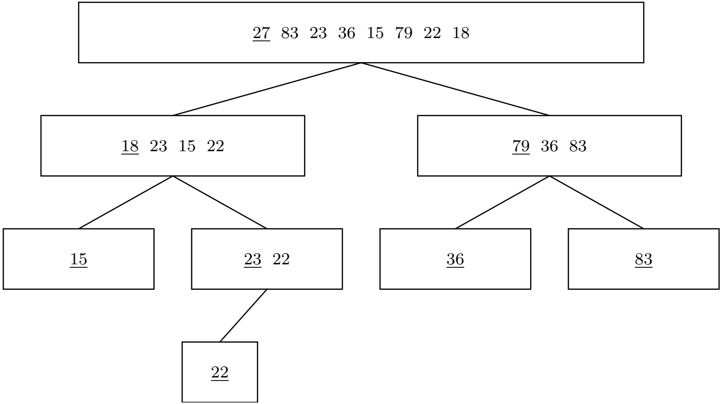
- The root is the top-level list (of all items to be sorted)
- The children of a node are the two sublists to be sorted.



Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.

- The root is the top-level list (of all items to be sorted)
- The children of a node are the two sublists to be sorted.
- Identify each list with its split value.



Worst-case Analysis of Quicksort

Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.

Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.
- The number of possible comparisons is

$$\binom{n}{2} = O(n^2)$$

Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.
- The number of possible comparisons is

$$\binom{n}{2} = O(n^2)$$

- Hence the worst-case number of comparisons performed by Quicksort when sorting n items is $O(n^2)$.

Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.
- The number of possible comparisons is

$$\binom{n}{2} = O(n^2)$$

- Hence the worst-case number of comparisons performed by Quicksort when sorting n items is $O(n^2)$.
- Question: Is there a better bound? Is it $o(n^2)$? Or is it $\Theta(n^2)$?

Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.
- The number of possible comparisons is

$$\binom{n}{2} = O(n^2)$$

- Hence the worst-case number of comparisons performed by Quicksort when sorting n items is $O(n^2)$.
- Question: Is there a better bound? Is it $o(n^2)$? Or is it $\Theta(n^2)$?
- Answer: The bound is tight. It is $\Theta(n^2)$.

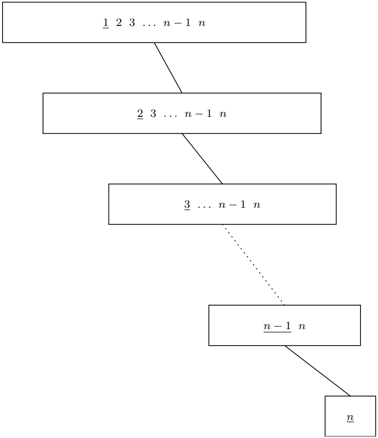
Worst-case Analysis of Quicksort

- Any pair of values x and y gets compared at most once during the entire run of Quicksort.
- The number of possible comparisons is

$$\binom{n}{2} = O(n^2)$$

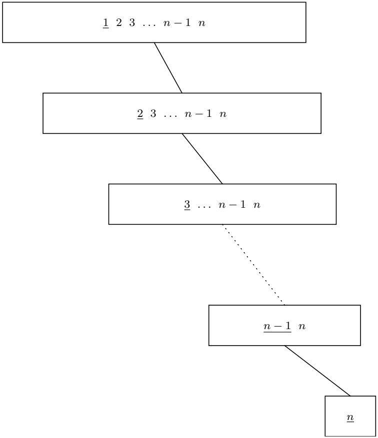
- Hence the worst-case number of comparisons performed by Quicksort when sorting n items is $O(n^2)$.
- Question: Is there a better bound? Is it $o(n^2)$? Or is it $\Theta(n^2)$?
- Answer: The bound is tight. It is $\Theta(n^2)$. We will see why on the next slide.

A bad case case for Quicksort: $1, 2, 3, \ldots, n-1, n$



$\binom{n}{2}$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$.

A bad case case for Quicksort: $1, 2, 3, \ldots, n-1, n$



$\binom{n}{2}$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$. But what about the average case ...?

Average-case analysis of Quicksort:

Our approach:

Average-case analysis of Quicksort:

Our approach:

1. Use the binary tree of sorted lists

Average-case analysis of Quicksort:

Our approach:

1. Use the binary tree of sorted lists
2. Number the items in sorted order

Average-case analysis of Quicksort:

Our approach:

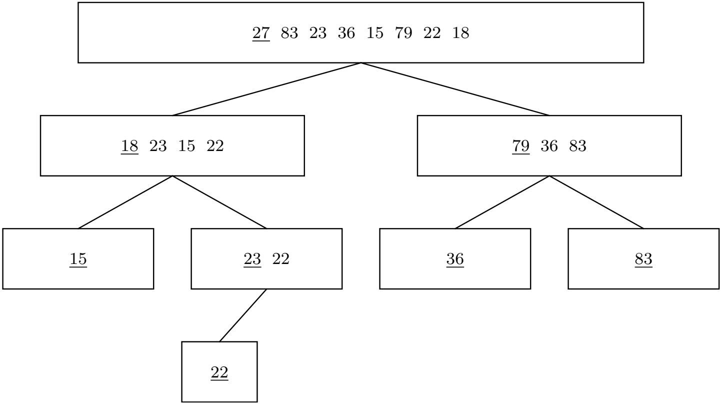
1. Use the binary tree of sorted lists
2. Number the items in sorted order
3. Calculate the probability that two items get compared

Average-case analysis of Quicksort:

Our approach:

1. Use the binary tree of sorted lists
2. Number the items in sorted order
3. Calculate the probability that two items get compared
4. Use this to compute the expected number of comparisons performed by Quicksort.

Average-case analysis of Quicksort:



Sorted order: | 15 18 22 23 27 36 79 83 |

Average-case analysis of Quicksort

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.
  - If S_i is chosen first, it is compared against all the other keys in the set in the split step (including S_j).

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.
  - If S_i is chosen first, it is compared against all the other keys in the set in the split step (including S_j).
  - Similar if S_j is chosen first.

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.
  - If S_i is chosen first, it is compared against all the other keys in the set in the split step (including S_j).
  - Similar if S_j is chosen first.

Examples:

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.
  - If S_i is chosen first, it is compared against all the other keys in the set in the split step (including S_j).
  - Similar if S_j is chosen first.

Examples:

- 23 and 22 (both statements true)

Average-case analysis of Quicksort

- Number the keys in sorted order: $S_1 < S_2 < \cdots < S_n$.
- Fact about comparisons: During the run of Quicksort, two keys S_i and S_j get compared if and only if the first key from the set of keys $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot is either S_i or S_j.
  - If some key S_k is chosen first with $S_i < S_k < S_j$, then S_i goes in the left half, S_j goes in the right half, and S_i and S_j never get compared.
  - If S_i is chosen first, it is compared against all the other keys in the set in the split step (including S_j).
  - Similar if S_j is chosen first.

Examples:

- 23 and 22 (both statements true)
- 36 and 83 (both statements false)

Average-case analysis of Quicksort

Average-case analysis of Quicksort

Assume:

Average-case analysis of Quicksort

Assume:

- All n keys are distinct

Average-case analysis of Quicksort

Assume:

- All n keys are distinct
- All permutations are equally likely

Average-case analysis of Quicksort

Assume:

- All n keys are distinct
- All permutations are equally likely
- The keys in sorted order are $S_1 < S_2 < \cdots < S_n$.

Average-case analysis of Quicksort

Assume:

- All n keys are distinct
- All permutations are equally likely
- The keys in sorted order are $S_1 < S_2 < \cdots < S_n$.

Let $P_{i,j} =$ The probability that keys S_i and S_j are compared with each other during the invocation of quicksort

Average-case analysis of Quicksort

Assume:

- All n keys are distinct
- All permutations are equally likely
- The keys in sorted order are $S_1 < S_2 < \cdots < S_n$.

Let $P_{i,j} =$ The probability that keys S_i and S_j are compared with each other during the invocation of quicksort

Then by Fact about comparisons on previous slide:

$P_{i,j}$ = The probability that the first key from $\{S_i, S_{i+1}, \ldots, S_j\}$ to be chosen as a pivot value is either S_i or S_j

Average-case analysis of Quicksort

Assume:

- All n keys are distinct
- All permutations are equally likely
- The keys in sorted order are $S_1 < S_2 < \cdots < S_n$.

Let $P_{i,j} =$ The probability that keys S_i and S_j are compared with each other during the invocation of quicksort

Then by Fact about comparisons on previous slide:

$$\begin{aligned} P_{i,j} &= \text{The probability that the first key from } \{S_i, S_{i+1}, \ldots, S_j\} \text{ to be chosen as a pivot value is either } S_i \text{ or } S_j \\ &= \frac{2}{j-i+1} \end{aligned}$$

Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do } \underline{\text{not}} \text{ get compared} \end{cases}$$

Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do } \underline{\text{not}} \text{ get compared} \end{cases}$$

1. The total number of comparisons is:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do } \underline{\text{not}} \text{ get compared} \end{cases}$$

1. The total number of comparisons is:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

2. The expected (average) total number of comparisons is:

$$E\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}\right) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right)$$

Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do } \underline{\text{not}} \text{ get compared} \end{cases}$$

1. The total number of comparisons is:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

2. The expected (average) total number of comparisons is:

$$E\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}\right) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right)$$

3. The expected value of $X_{i,j}$ is:

$$E(X_{i,j}) = P_{i,j} = \frac{2}{j-i+1}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right)$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) \quad = \quad \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{2}{j-i+1}\\ &= \sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k} \quad (k=j-i+1)\end{aligned}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k}
\end{aligned}
$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E(X_{i,j}) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j-i+1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2 \sum_{i=1}^{n} H_n
\end{aligned}
$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2 \sum_{i=1}^{n} H_n = 2nH_n
\end{aligned}
$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \qquad (k = j - i + 1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2 \sum_{i=1}^{n} H_n = 2nH_n \in O(n \lg n).
\end{aligned}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2 \sum_{i=1}^{n} H_n = 2nH_n \in O(n \lg n).
\end{aligned}$$

Average-case analysis of Quicksort

Hence the expected number of comparisons is

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j-i+1) \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \\
&= 2 \sum_{i=1}^{n} H_n = 2nH_n \in O(n \lg n).
\end{aligned}
$$

So the average time for Quicksort is $O(n \lg n)$.